3. PROBABILITY DISTRIBUTIONS: $P_1(\mathcal{E}>\mathcal{E}_0)_{A,B}$, $(0 < \alpha < 2)$

We are now ready to obtain the first-order exceedance probabilities $P_1(E > E_0)$, cf. (2.22b), when an independent gaussian component is present, so that Eqs. (2.78), and (2.90), (2.93) apply respectively for the characteristic functions for Class A and B interference, to be used in (2.22b).

First, however, it is convenient to introduce the following normalizations:

$$\underline{\text{Class A:}}\quad \mathcal{E}_0 \equiv E_0/\sqrt{2\Omega_{2A}(1+\Gamma_A')} \quad ; \quad \mathcal{E} = E/\sqrt{2\Omega_{2A}(1+\Gamma_A')} \tag{3.1}$$

with

$$\Omega_{2A} \equiv A_{\infty,A}\left\langle\hat{B}^2_{0,A}\right\rangle/2 \quad ; \quad \Gamma_A' \equiv \sigma_G^2/\Omega_{2A}: \ \frac{\text{gauss intensity}}{\text{"impulsive" intensity}} \ . \tag{3.1a}$$

For Class B noise we use (5.14), viz.,

$$\underline{\text{Class B:}}\quad \mathcal{E}_0 \equiv \frac{E_0}{\sqrt{2\Omega_{2B}(1+\Gamma_B')}} \quad ; \quad \mathcal{E} \equiv \frac{E}{\sqrt{2\Omega_{2B}(1+\Gamma_B')}} \ , \tag{3.2}$$

where, cf. (5.14a), we have

$$\Omega_{2B} \equiv \left\{\frac{A_{\infty,B}\left\langle\hat{B}^2_{0,B}\right\rangle}{2}\right\} \quad ; \quad \Gamma_B' \equiv \sigma_G^2/\Omega_{2B} \tag{3.2a}$$

which are directly analogous to the corresponding parameters above for Class A noise. Then, writing

$$a_{A\,or\,B} \equiv \left\{2\Omega_2(1+\Gamma')\right\}^{-1/2}, \ \text{A or B} \ , \tag{3.3}$$

we see that $\mathcal{E} = aE$, $\mathcal{E}_0 = aE_0$ in each case, and $\therefore$ $r = a\lambda$ in (2.78), (2.90) and (2.93), so that the desired exceedance probabilities now have the generic form

$$P_1(\mathcal{E} > \mathcal{E}_0)_{A,B} = 1-\mathcal{E}_0\int_0^{\infty} J_1(\lambda\mathcal{E}_0)\hat{F}_1(ia\lambda)_{A,B}d\lambda, \tag{3.4}$$

cf. (2.22b), where, of course, the specific parameter values in the normalization factor $\underline{a}$ have different forms for the Class A and B interference.

3.1 Class A Interference:

Applying (3.1) - (3.3) to (2.78) and omitting the "correction terms" allows us to write the Class A c.f. in the following desired approximate form:

$$\hat{F}_1(ia\lambda)_A \doteq e^{-A_A} \sum_{m=0}^{\infty} \frac{A_A^m}{m!} e^{-\hat{\sigma}_{mA}^2 a^2 \lambda^2/2} \; ; \quad 2\hat{\sigma}_{mA}^2 \equiv (m/A_A+\Gamma_A')/(1+\Gamma_A') \tag{3.5}$$

cf. Eq. (3.17)* [Middleton, 1974], and we henceforth abbreviate $A_{\infty,A} = A_A$, etc. Applying (3.5) to (3.4) then gives us

$$P_1(\mathcal{E}>\mathcal{E}_0)_A \simeq 1 - \mathcal{E}_0 e^{-A_A} \sum_{m=0}^{\infty} \frac{A_A^m}{m!} \int_0^{\infty} J_1(\mathcal{E}_0\lambda) e^{-\hat{\sigma}_{mA}^2 a^2 \lambda^2/2} d\lambda, \tag{3.6}$$

which with the help of (2.25) and the relation ${}_1F_1(1;2;-x) = (1-e^{-x})/x$, [Eq. A.1-19b, Middleton, 1960], becomes**

$$\boxed{\left.\begin{aligned} P_1(\mathcal{E}>\mathcal{E}_0)_A &\simeq 1-e^{-A_A} \sum_{m=0}^{\infty} \frac{A_A^m}{m!} \frac{\mathcal{E}_0^2}{2\hat{\sigma}_{mA}^2} \, {}_1F_1(1;2;-\mathcal{E}_0^2/2\hat{\sigma}_{mA}^2) && (3.7a) \\ &\simeq e^{-A_A} \sum_{m=0}^{\infty} \frac{A_A^m}{m!} e^{-\mathcal{E}_0^2/2\hat{\sigma}_{mA}^2} && (3.7b) \end{aligned}\right\} \mathcal{E}_0 \geq 0 \, ,}$$

------

* Note that $2\hat{\sigma}_{mA}^2$ here is equal to σ_{mA}^2, cf. Eq. (5.7), of Middleton (1974).

** This PD is properly normalized for $\langle \mathcal{E}^2 \rangle_A = 1$, cf. remarks following Eq. (2.78).

for the desired approximation* of P_{1-A}. We observe at once that the Class A exceedance probability P_{1-A} is (primarily) a weighted sum of rayleigh probability distributions (P.D.'s), each with a variance which increases with order (m). Note from (3.7) that

$$P_1(\mathcal{E}\geq 0)_A = 1; \quad P_1(\mathcal{E}>\mathcal{E}_o\to\infty)_A = 0 ;$$

$$P_1(\mathcal{E}>\mathcal{E}_o\doteq 0)_A \doteq 1-\left(e^{-A_A}\sum_{m=0}^{\infty}\frac{(1+\Gamma_A')\,A_A^m}{(\Gamma_A'+m/A_A)\,m!}\right)\mathcal{E}_o^2 \quad , \tag{3.8}$$

and since each term of (3.7b) is positive and the exponential less than unity, $0 \leq P_{1-A} \leq 1$, also with $P_{1-A}^{\cdot}$ monotonically decreasing as $\mathcal{E}_o \to \infty$, all as required for a proper (exceedance) P.D. Furthermore, the expected "rayleigh"-form of the P.D. is exhibited in (3.8) for small thresholds, e.g.,

$$P_1(\mathcal{E}>\mathcal{E}_o)_A \doteq 1-B_A\mathcal{E}_o^2 \doteq e^{-B_A\mathcal{E}_o^2} \quad ; \quad B_A\mathcal{E}_o^2 << 1, \tag{3.9a}$$

where explicitly now

$$B_A \equiv (1+\Gamma_A')e^{-A_A}\sum_{m=0}^{\infty}\frac{A_A^{m+1}}{m!}\,\frac{1}{(m+A_A\Gamma_A')} \quad ,$$

which depends, of course, on the Impulsive Index A_A and on the intensity ratio Γ_A', cf. (3.1a).

---

* The correction terms (containing $\hat{C}_4,\hat{C}_6$, etc.) in the c.f., and hence in the P.D., may become important for extremely large $\mathcal{E}_o$ and very small values of the Index A_A, although present experimental results, and theory, indicate that the principal effects for large values of $\mathcal{E}_o$ are satisfactorily accounted for by the approximation (3.7). We reserve to a later study the investigation of these effects.

Figs. 3.1II and 3.2II, based on (3.7) show some typical distributions P_{1-A} vs. threshold $\mathcal{E}_0$, with Γ'_A and A_A respectively as parameters. As expected, these PD's are highly nonrayleigh* for the rarer "events", e.g. those which exceed the larger thresholds $\mathcal{E}_0$, while the rayleigh forms appear for the less rare events ($\mathcal{E}_0$ small), also as expected, cf. (3.9). Thus while the slope $(dP_1/d\mathcal{E}_0)$ is a constant -2 ($\equiv e^{-x^2}$) for the small amplitudes on these log vs. log^2 probability scales, it is an (approximate) -1.2 for $\Gamma' = 10^{-4}$, $A_A = 10^{-1}$ for $\mathcal{E}_0 \to \infty$, i.e. a fall-off ($\equiv e^{-x^{(1.2)}}$ of P_{1-A} at large $\mathcal{E}_0$ somewhat faster than exponential, which latter is consistent with the required existence of all moments, cf. Sec. 5. Different values of A_A, Γ'_A lead to different limiting slopes as $\mathcal{E}_0 \to \infty$, but all are dominated by the exponential type of fall-off. In addition, as the relative size of the gaussian component increases (increasing Γ'_A) so does the threshold $\mathcal{E}_0$ rise, above which the nonrayleigh effects appear. Similarly, as the Impulsive Index A_A increases, i.e. the envelope distribution approaches eventually the limiting rayleigh form (2.57b) [with (2.53a)] as $A_A \to \infty$, the very steep "neck" of the curve becomes less extensive and shifts to the larger probabilities (lower $\mathcal{E}_0$), also as expected. In the limit $A_A \to \infty$ this "neck" disappears entirely and the straight-line (slope-2) rayleigh distribution appears, for all $\mathcal{E}_0$. [Development of these numerical results to a much more extensive and fine grid of parameter values is planned for a later Report.]

3.2 Class B Interference ($0 < \alpha < 2$):

Applying the normalizations (3.2)-(3.3) to (2.90), (2.93) now, for the two c.f.'s which approximate the Class B interference, we obtain explicitly**

---

* For envelopes E here, e.g., equivalently nongaussian for the corresponding instantaneous amplitudes X [Middleton, 1974].

** For compactness, we set $A_{\infty,B} \equiv A_B$, henceforth, cf. (3.5) et seq. above .

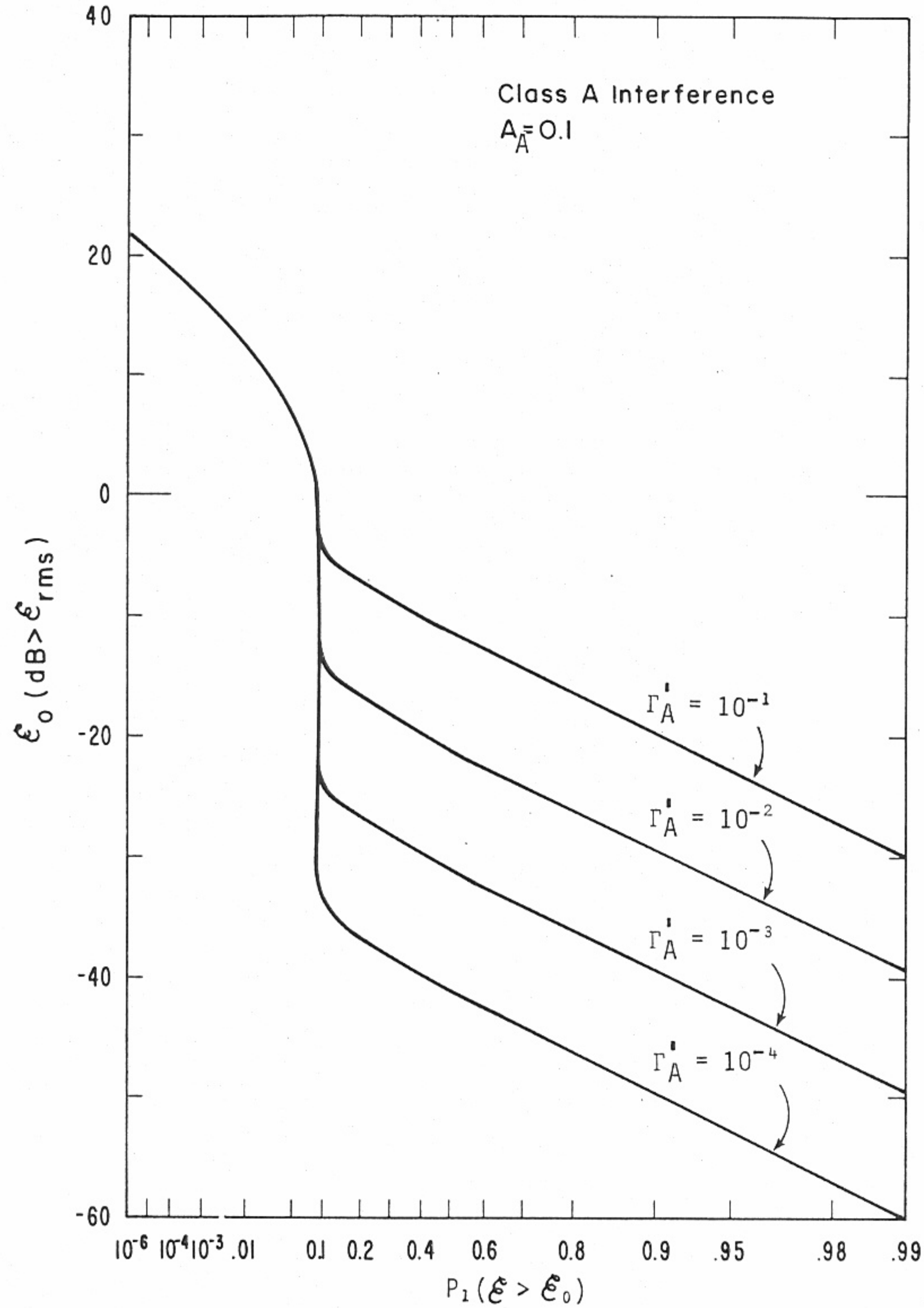

Figure 3.1 (II). The envelope distribution $[\mathrm{Prob}(\mathcal{E} > \mathcal{E}_0)]$ calculated for Class A interference for $A_A = 0.1$ and various Γ_A' from eq. (3.7b).

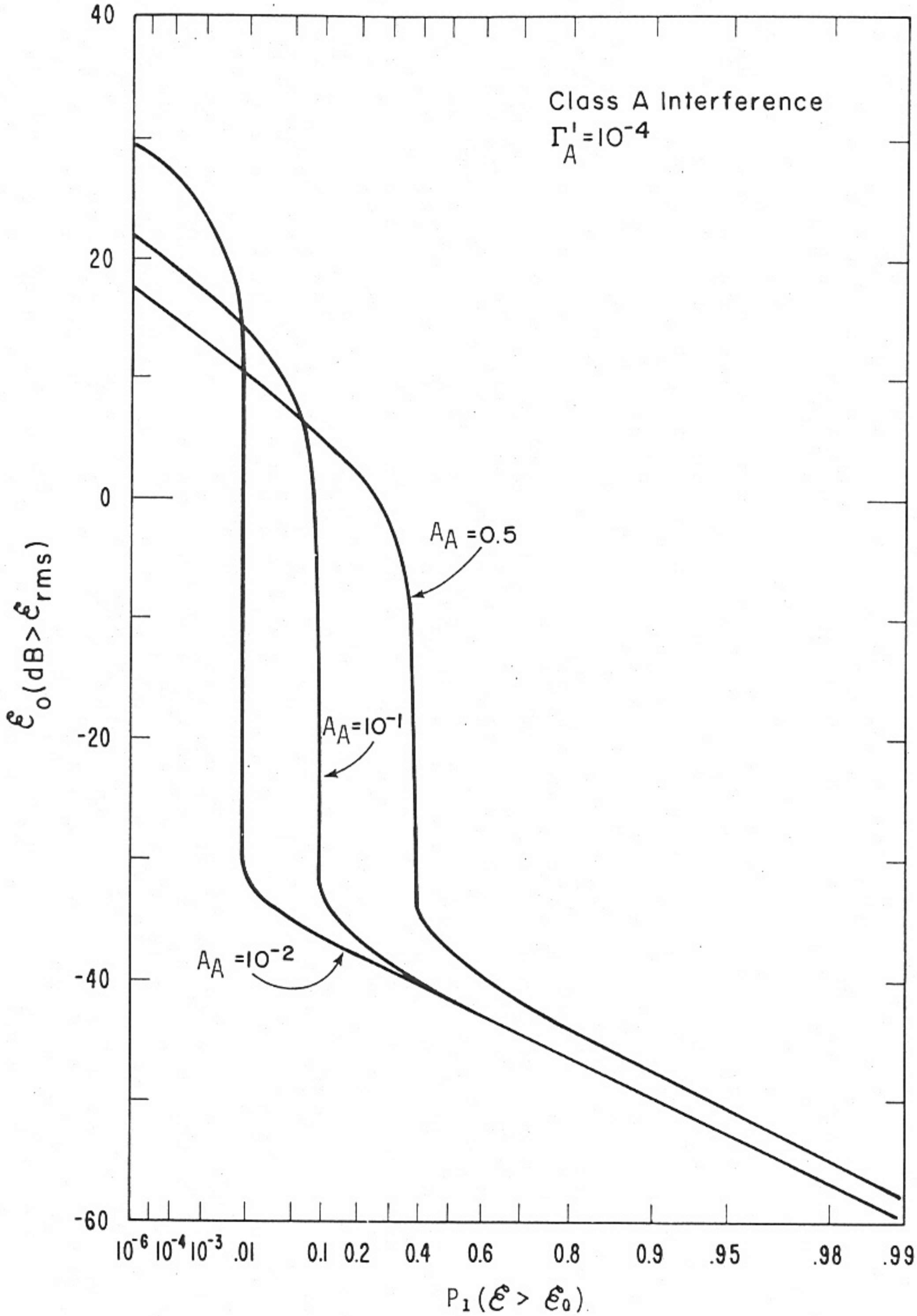

Figure 3.2 (II). The envelope distribution $[\mathrm{Prob}(\mathcal{E} > \mathcal{E}_0)]$ calculated for Class A interference for $\Gamma_A' = 10^{-4}$ and various A_A from eq. (3.7b).

$$\hat{F}_1(ia\lambda)_{B-I} \doteq e^{-b_{1\alpha}A_B a^\alpha \lambda^\alpha - \Delta\sigma_G^2 a^2\lambda^2/2} \quad , \tag{3.10a}$$

$$\hat{F}_1(ia\lambda)_{B-II} \doteq e^{-A_B}\cdot\exp[A_B e^{-b_{2\alpha}a^2\lambda^2/2} - \sigma_G^2 a^2\lambda^2/2]. \tag{3.10b}$$

These, in turn are applied to (3.4) to give us $P_{1-I,II}$ respectively. Starting with (3.10a), we expand the exponential in λ^α and use (2.55) to get

$$P_1(\mathcal{E} > \mathcal{E}_0)_{B-I} \simeq 1-\mathcal{E}_0 \sum_{n=0}^{\infty} \frac{(b_{1\alpha}A_B a^\alpha)^n}{n!} (-1)^n \int_0^\infty \lambda^{n\alpha} J_1(\lambda\mathcal{E}_0) e^{-\Delta\sigma_G^2 a^2\lambda^2/2} d\lambda \tag{3.11a}$$

$$\equiv \hat{P}_1(\hat{\mathcal{E}}>\hat{\mathcal{E}}_0) \simeq 1-\hat{\mathcal{E}}_0^2 \sum_{n=0}^{\infty} \frac{(-1)^n \hat{A}_\alpha^n}{n!} \Gamma(1+\tfrac{\alpha n}{2})\, {}_1F_1(1+\tfrac{\alpha n}{2};\, 2;\, -\hat{\mathcal{E}}_0^2) \ , \tag{3.11b}$$

with

$$\hat{\mathcal{E}}_0 \equiv (\mathcal{E}_0 N_I)/2G_B;\ \hat{A}_\alpha \equiv A_\alpha/2^\alpha G_B^\alpha \ , \qquad [\mathcal{E}_0 \to \mathcal{E}_0 N_I \text{ in (3.11a)}], \tag{3.11c}$$

$$A_\alpha \equiv 2^\alpha b_{1\alpha} a^\alpha A_B = 2^\alpha b_{1\alpha} A_B/[2\Omega_{2B}(1+\Gamma_B')]^{\alpha/2} = \frac{2\Gamma(1-\alpha/2)}{\Gamma(1+\ /2)} A_B \left\langle \left(\frac{\hat{B}_{0,B}}{\sqrt{2\Omega_{2B}(1+\Gamma_B')}} \right)^\alpha \right\rangle \tag{3.12a}$$

and

$$G_B^2 = \tfrac{1}{4}(1+\Gamma_B')^{-1}\left(\tfrac{4-\alpha}{2-\alpha} + \Gamma_B'\right) \ , \tag{3.12b}$$

cf. (2.88a,b,c), (3.3a), where N_I is a <u>scaling</u> factor which scales $P_{1-(B-I)}$, $w_{1-(B-I)}$ to insure that $\langle \mathcal{E}^2 \rangle_B = 1$, cf. (2.94) ff, where $a_B^2\Delta\sigma_G^2 \equiv 2G_B^2$. The quantity $\hat{A}_\alpha$ is the <u>Effective Class B Impulsive Index</u>, which is proportional to the Impulsive Index A_B, for this Class B interference. In addition, it depends spatially on the spatially sensitive parameter, α, and on the relative gauss component Γ_B', (3.2a).

With the help of Kummer's transformation [Middleton, 1960, Section A.1.2, p. 1073, Eq. A.1-17] we can write (3.11b) alternatively as

$$\hat{P}_1(\hat{\mathcal{E}} > \hat{\mathcal{E}}_0)_{B-I} \simeq e^{-\hat{\mathcal{E}}_0^2}\left\{1-\hat{\mathcal{E}}_0^2 \sum_{n=1}^{\infty} \frac{(-1)^n \hat{A}_\alpha^n}{n!} \Gamma(1+\tfrac{\alpha n}{2})\, {}_1F_1(1-\tfrac{\alpha n}{2}; 2; \hat{\mathcal{E}}_0^2)\right\} , \tag{3.13}$$

where we have used Eq. (A.1-19b) [Middleton, 1960]. For large $\mathcal{E}_0$ we obtain formally, with the help of the asymptotic relation [Middleton, 1960, (A.1-16b), p. 1073],

$$_1F_1(\alpha;\beta;-x) \simeq \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} x^{-\alpha}\left[1+\frac{\alpha(\alpha-\beta+1)}{1!x}+\frac{\alpha(\alpha+1)(\alpha-\beta+1)(\alpha-\beta+2)}{2!x^2}+\dots\right], \tag{3.14}$$

the following expression for P_1:

$$\hat{P}_1(\hat{\mathcal{E}}>\hat{\mathcal{E}}_0)_{B-I} \simeq \sum_{n=1}^{\infty}\frac{\hat{A}_\alpha^n(-1)^{n+1}}{n!}\ \frac{\Gamma(1+\frac{\alpha n}{2})}{\Gamma(1-\frac{\alpha n}{2})}\hat{\mathcal{E}}_0^{-n\alpha}\left[1+\frac{(1+\alpha n/2)(\alpha n)}{2\hat{\mathcal{E}}_0^2}+\dots\right],$$
$$\mathcal{E}_0^2 \gg 1. \tag{3.15}$$

This shows that $\lim_{\mathcal{E}_0\to\infty} P_{1-I} \to O(\mathcal{E}_0^{-\alpha}) \to 0$. However, as explained in A, Section 2.7, for $\mathcal{E}_0$ greater than some (large) value $\mathcal{E}_B$, which is determined from Eq. (3.19e) below, we must use the second form of c.f., (3.10b). Figs.3.3II,3.4II here are based on (3.11b), (3.15), and are valid representations, provided $\mathcal{E}_0$ is not too large, e.g. $\mathcal{E}_0 \leq \mathcal{E}_B$.

For the "rare events", or large $\mathcal{E}_0$, we apply (3.10b) to (3.4), as discussed earlier (cf. A, Sec. 2.7), to obtain

$$P_1(\mathcal{E}>\mathcal{E}_0)_{V-II} \simeq 1-\mathcal{E}_0 e^{-A_B}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\int_0^\infty J_1(\mathcal{E}_0\lambda)e^{-\hat{\sigma}_{mB}^2 a^2\lambda^2/2}\,d\lambda, \tag{3.16}$$

with

$$2\hat{\sigma}_{mB}^2 = 2(mb_{2\alpha}+\sigma_G^2)a_B^2 = \left(\frac{m}{\hat{A}_B}+\Gamma_B'\right)/(1+\Gamma_B')\ ,\quad \hat{A}_B \equiv A_B\left(\frac{2-\alpha}{4-\alpha}\right) \tag{3.16a}$$

from (3.2a),(2.88c), cf.(3.5), (3.6): thus $\hat{\sigma}_{mB}$ has the same <u>form</u> as $\hat{\sigma}_{mA}$, (3.6). Accordingly, we may use the result (3.7b), rewriting it here for this large-magnitude approximation for Class B noise, as*

$$\boxed{P_1(\mathcal{E}>\mathcal{E}_0)_{B-II} \simeq \frac{e^{-A_B}}{4G_B^2}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}e^{-\mathcal{E}_0^2/2\hat{\sigma}_{mB}^2}\ ,\quad (\mathcal{E}_0>\mathcal{E}_B)\ .} \tag{3.17}$$

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* This PD is now properly normalized [remarks after (2.94) and], cf. (i), (3.17a,b), and discussion following.

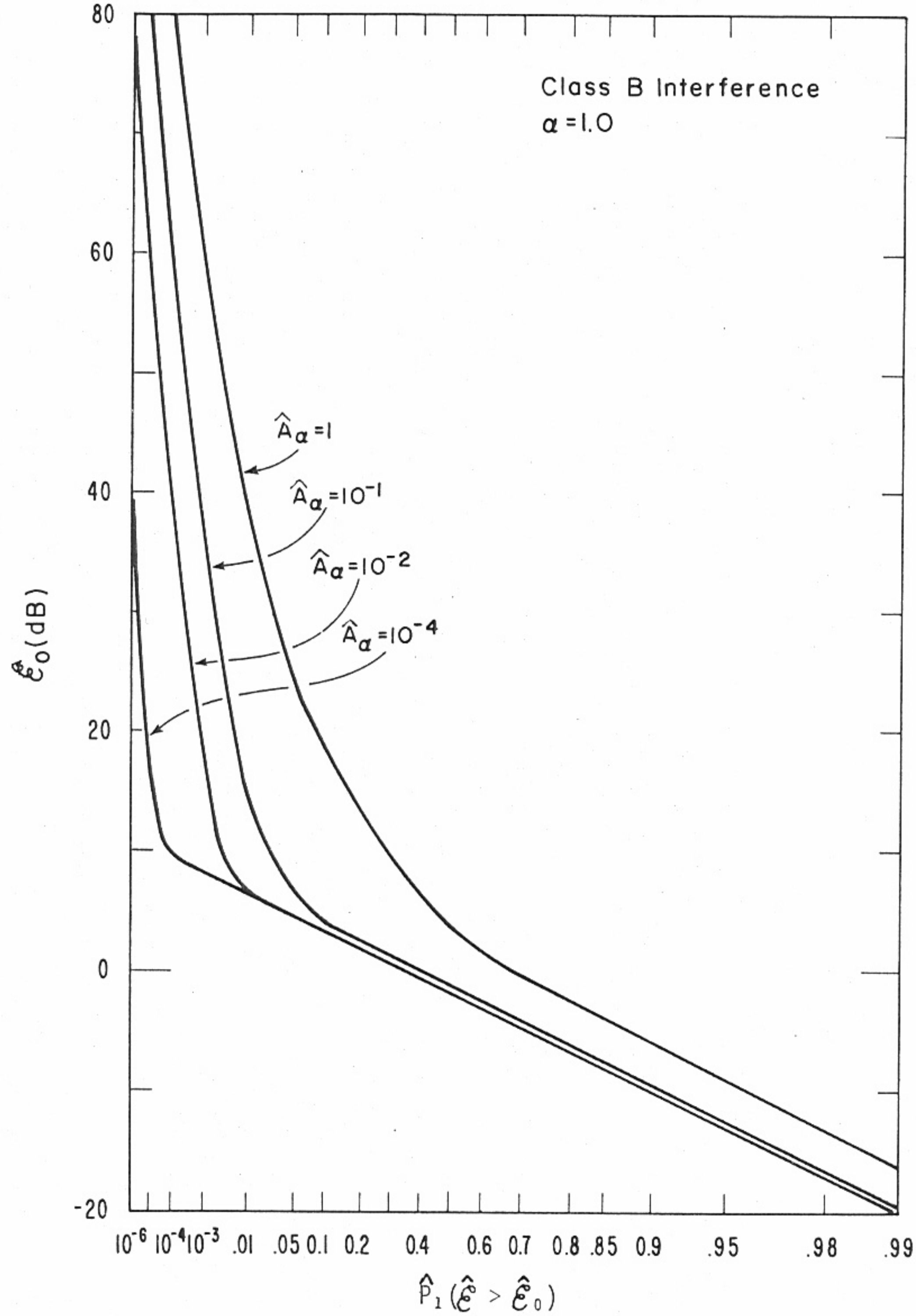

Figure 3.3 (II). The envelope distribution $[\mathrm{Prob}(\hat{\mathcal{E}} > \hat{\mathcal{E}}_0)]$ calculated for Class B interference for $\alpha = 1.0$ for various $\hat{A}_\alpha$ from eqs. (3.11b, 3.15).

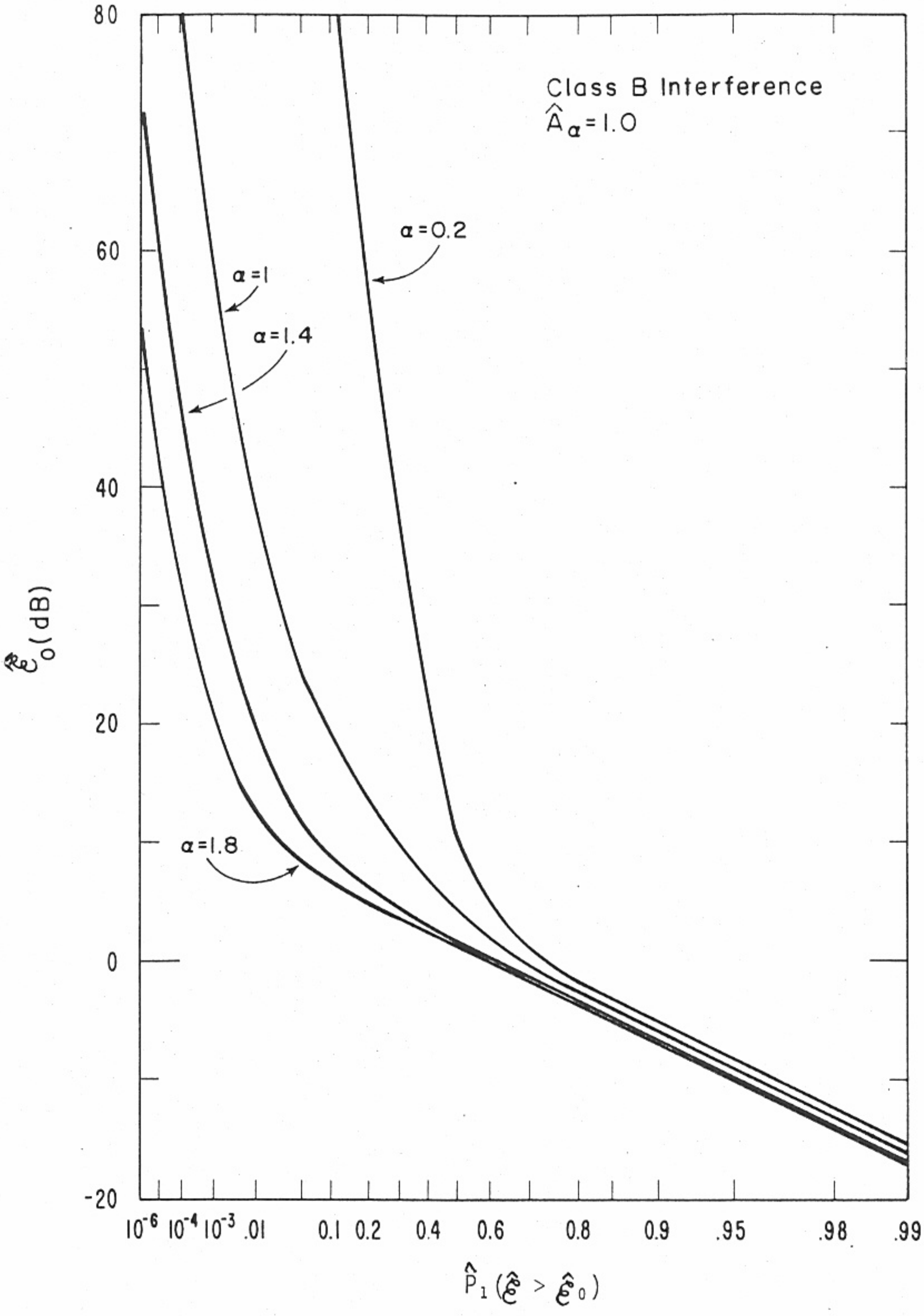

Figure 3.4 (II). The envelope distribution $[\mathrm{Prob}(\hat{\mathcal{E}} > \hat{\mathcal{E}}_0)]$ calculated for Class B interference for $\hat{A}_\alpha = 1.0$ for various α from eqs. (3.11b, 3.15).

[Figures 3.1II, 3.2II for Class A interference illustrate the character of (3.17), which, of course, is only applicable here, Class B, for the larger values of $\mathcal{E}_o$ ($>\mathcal{E}_B$).]

A. The Composite Approximation:

The problem with the approximating results for P_{1-B} in the Case B model, cf. (3.11b) and (3.17), is that these forms, stemming as they do from approximate c.f.'s [cf. (3.10a,b)], are not properly scaled, or "normalized", in the sense that each approximating form, P_{1-I}, P_{1-II}, does not yield the correct mean square value of $\langle\mathcal{E}^2\rangle_B = 1$ or $\langle E^2\rangle_B = 2\Omega_{2B}(1+\Gamma_B')$, cf. (5.14) with (3.2), and the remarks following Eq. (2.94) above. The approximation P_{1-I}, and its associated pdf, w_{1-I}, (4.3), in fact, do not possess a finite mean square on $(0,\infty)$, cf. Sec. (5.3)ff., while P_{1-II}, the "Type A" form and its pdf., w_{1-II}, (4.4), yields $\langle\mathcal{E}^2\rangle_{B-II} \neq 1$.

Accordingly, since the precise mean square is finite and is known to be $\langle\mathcal{E}^2\rangle_B = 1$, by calculation from the exact c.f. [cf. (5.10a), and Section 5.2-B], we must suitably scale (or "normalize") w_{1-I}, w_{1-II} (4.5) so that $\langle\mathcal{E}^2\rangle_B$, cf. (5.6c), exists and is equal to unity. This is done as follows:

(i). Let us consider first w_{1-II}, (4.4), and calculate $\langle\mathcal{E}^2\rangle_{II}$ on $(0<\mathcal{E}<\infty)$ according to (5.1). The result is easily seen to be

$$\langle\mathcal{E}^2\rangle_{B-II} = e^{-A_B} \sum_{m=0}^{\infty} \frac{(m/\hat{A}_B+\Gamma_B')}{(1+\Gamma_B')m!} A_B^m = \frac{\frac{4-\alpha}{2-\alpha}+\Gamma_B'}{1+\Gamma_B'} = 4G_B^2(\neq 1), \tag{3.17a}$$

where G_B^2 is given by (3.12b), so that here we require the normalization factor $N_{II}^2 = (4G_B^2)^{-1}$, e.g.

$$w_1(\mathcal{E})_{B-II-norm} = \frac{1}{4G_B^2} w_1(\mathcal{E})_{B-II} = N_{II}^2 w_1(\mathcal{E})_{B-II}. \tag{3.17b}$$

(Henceforth in the text we write $w_1(\mathcal{E})_{B-II}, P_{1-II}$ in normalized form, which are then used for anlytical and numerical calculations in the remaining sections of Part II here.)

(ii). The case of w_{1-I}, (4.3), requires a different approach, since $\langle\mathcal{E}^2\rangle_{B-I}$ on $(0<\mathcal{E}<\infty)$ becomes infinite $(0<\alpha<2)$, cf. Section 5.3: [$\langle\mathcal{E}^2\rangle_{B-II}$ on

$(0<\mathcal{E}\leq\mathcal{E}_B<\infty)$, of course, is finite, cf. (5.6c)]. Here we need to scale $\mathcal{E}_0$ according to (3.11b) above: $\underline{\mathcal{E}_0 \to \mathcal{E}_0 N_I}$ (and $\therefore \hat{\mathcal{E}}_0 = (\mathcal{E}_0 N_I)/2G_B$). The rationale for this is the observation that P_{1-I} (and w_{1-I}) must have the same values in the rayleigh region ($\mathcal{E}_0^2<<1$), where $P_{1-I} \sim 0.9$, or 0.99, etc., as does the precise distribution, P_{1-B}, based on the (intractable but) exact c.f. (2.87), hence (3.11b). The scaling factor, N_I, is to be determined by fitting the two approximate forms P_{1-I}, P_{1-II} together by the procedure outlined below, which is based on the canonical properties of the Class B model generally. Note, finally, that the "Class A" form (II) is coupled to the Class B form (I) through the Class B parameter α, and vice versa through the "Class A" parameter Γ_B', appearing in G_B, common to both approximations I,II.

To combine the suitably scaled P_{1-I} and normalized P_{1-II} to form the composite approximation for Class B interference which is valid for all $\mathcal{E}_0 \geq 0$ we now use the following desired properties of $P_{1\text{-composite}}$, which is sketched in Fig. 3.5II:

(i). $P_{1-I} = P_{1-II}$ in the rayleigh region, e.g. $0 \leq \mathcal{E}_0$ small. Equality of the two approximations in the rayleigh region is required, since both must represent the same (small) amplitude behaviour, characteristic of all these PD's.

(ii). $\dfrac{dP_{1-I}}{d\mathcal{E}_0} = \dfrac{dP_{1-II}}{d\mathcal{E}_0}$ in the rayleigh region.

(iii). $P_{1-I} = P_{1-II}$ at the "bendover" or junction point $\mathcal{E}_B$ of the two approximations, cf. Fig.(3.5)II. This point, $\mathcal{E}_B$, is empirically determined from the data, e.g. from the experimental APD or exceedance probability curve $P_1(\mathcal{E} > \mathcal{E}_0)_{exp.}$, as described below, cf. (vi).

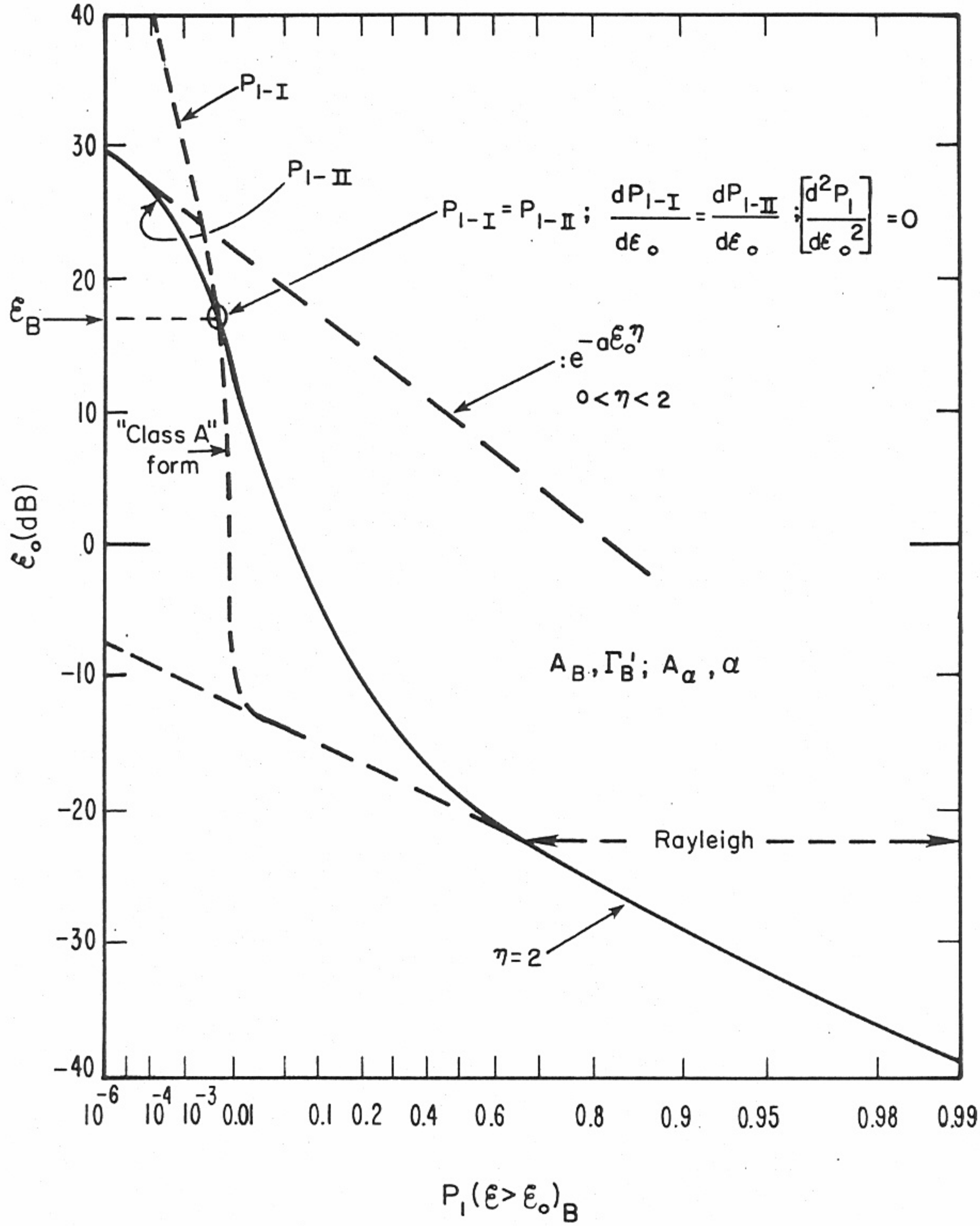

Figure 3.5 (II). Schema of P_{1-B}, eq. (3.20), obtained by joining the two approximating forms (3.11b, 3.17).

(iv). $\left(\frac{dP_{1-I}}{d\mathcal{E}} = \frac{dP_{1-II}}{d\mathcal{E}}\right)_{\mathcal{E}_B}$: the (finite) slopes of the approximating $P_{1-I,II}$ are equal at $\mathcal{E}_B$: this insures a common tangent, i.e., a smooth fit; moreover, we have

(v). $\left(\frac{d^2P_{1-I}}{d\mathcal{E}^2} = \frac{d^2P_{1-II}}{d\mathcal{E}^2}\right)_{\mathcal{E}_B}$ $(\neq 0)$: this follows as a consequence of (iv), and the continuity of the derivative at $\mathcal{E}_B$, insuring that the associated pdf's are continuous at the joining point $\mathcal{E}_B$. However, note that $\mathcal{E}_B$ is not usually a point of inflexion of $P_{1-I,II}$.

(vi). $\mathcal{E}_B$: this is the point of inflexion $(d^2P_1/d\mathcal{E}_B^2 = 0)$ of the actual P_1, and is determined as such (by inspection, usually), of the empirical exceedance probability $P_{1-exp.}$, cf. Fig. (3.5)II. (3.18)

Accordingly, from (3.11b), (3.15), (3.17) we have explicitly for (i)-(v) above:

$$\text{(i).}\quad \frac{\mathcal{E}_o^2N_I^2}{4G_B^2}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{A_\alpha}{2^\alpha G_B^\alpha}\right)^n}{n!}\Gamma\left(1+\frac{\alpha n}{2}\right) \doteq \frac{\mathcal{E}_o^2e^{-A_B}}{4G_B^2}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}\left(2\hat{\sigma}_{mB}^2\right)^{-1}\;; \tag{3.19a}$$

[(ii). [Same as (3.19a), without the $2\mathcal{E}_0$ factors common to both members of the equation; however, (ii) is here implied by the <u>form</u> of (i) and does not provide new information.] (3.19b)

$$\text{(iii).}\quad \frac{A_\alpha\Gamma(1+\alpha/2)}{2^\alpha G_B^\alpha\Gamma(1-\alpha/2)}\left(\frac{\mathcal{E}_BN_I}{2G_B}\right)^{-\alpha}[1+O_{(iii)}([\mathcal{E}_BN_I]^{-\alpha};\mathcal{E}_B^2N_I^2)] = \frac{e^{-A_B}}{4G_B^2}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}e^{-\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2}\;;$$

$$\text{(iv).}\quad \frac{A_\alpha\alpha\Gamma(1+\alpha/2)}{2^\alpha G_B^\alpha\Gamma(1-\alpha/2)}\left(\frac{\mathcal{E}_BN_I}{2G_B}\right)^{-\alpha-1}[1+O_{(iv)}([\mathcal{E}_BN_I]^{-\alpha}[\mathcal{E}_BN_I]^3)] \simeq \frac{\mathcal{E}_Be^{-A_B}}{4G_B^2}\sum_{m=0}^{\infty}\frac{A_B^m}{m!}e^{-\mathcal{E}_B^2/2\hat{\sigma}_{mB}^2}\;; \tag{3.19d}$$

(v),(vi): $\mathcal{E}_B$ cannot be determined analytically from either approximating form $P_{1-I,II}$. It must be established as a point of inflexion from the <u>empirical</u> PD, as noted above. (3.19e)

[In using (iii), (iv), we may need at least the next set of "correction" terms in the asymptotic developments of P_{1-I}, $dP_{1-I}/d\mathcal{E}_B$.] We note that (given $\mathcal{E}_B$) these three relations [(i),(iii),(iv)] are sufficient to determine in principle, any three of the six parameters $N_I,\alpha,A_\alpha,A_B,\Omega_{2B},\Gamma'_B$ (cf. (3.16a)), when the other three are specified. Later, in Section 6, we shall show how (3.18), (3.19) may be extended to permit us to obtain, from the experimental exceedance probability $P_1(\mathcal{E}>\mathcal{E}_0)_{xpt}$, the six parameters $N_I,\alpha,A_\alpha,A_B,\Omega_{2B},\Gamma'_B$ (or, more fundamentally, $[\alpha,A_B,\langle\hat{B}^\alpha_{o,B}\rangle,\Omega_{2B},\Gamma'_B]$, cf. (3.12)).

For the illustrative calculations of Figs. 3.6II, 3.7II, it is convenient to preset $\mathcal{E}_B;N_I,\alpha,A_\alpha$, and determine $A_B,\Gamma'_B,\Omega_{2B}$ from (i),(iii),iiv). Other possibilities are: Fix $(\mathcal{E}_B;N_I,\alpha,\Gamma'_B)$, determine $(A_\alpha,\Omega_{2B},A_B)$; fix $(\mathcal{E}_B;A_\alpha,\alpha,\Gamma'_B)$, determine (N_I,A_B,Ω_{2B}); fix $(\mathcal{E}_B;A_B,\Gamma'_B,\Omega_{2B})$, determine (N_I,α,A_α); fix $(\mathcal{E}_B;N_I,A_B,\Gamma'_B)$, determine $(A_\alpha,\alpha,\Omega_{2B})$ etc. In any case, we have now

$$P_{1-B} = P_{1-I},\ 0 \le \mathcal{E}_0 \le \mathcal{E}_B;\ = P_{1-II},\ \mathcal{E}_0 \ge \mathcal{E}_B, \tag{3.20}$$

with P_{1-I}, P_{1-II} given respectively by (3.11b) and (3.17). The curves of Figs. 3.6II, 3.7II are equivalent to Figs. 3.1.1, 3.1.2 of Furutsu and Ishida [1960], with $(\nu/\alpha)_{F+I} \to A_B$, $(a_0/\sigma)_{F+I} \to (\Gamma'_B)^{-1}$, and $(R/\sigma)_{F+I} \to \mathcal{E}_0$ and exhibit the same kind of "elbow" in the transition region from the rayleigh behaviour (for $\mathcal{E}_0^2<<1$), with a bend-over to a constant slope ($P_1 \sim e^{-a\mathcal{E}_0^\eta}$, $\eta>0$), as for Class A noise, when $\mathcal{E}_0 \to \infty$, cf. Figs.3.1II,3.2II, 3.5II.

B. Remarks on Hall-Type Models:

Finally, we observe that a Hall model [Hall, 1966] may be obtained formally from the P_{1-I} form for the rayleigh and intermediate region ($0\le\mathcal{E}_0\le\mathcal{E}_B$), provided we neglect the gaussian contributions (e.g. $\Delta\sigma_G^2 \to 0$), so that the c.f. (2.89) now applies. From (2.89) in (3.4) we accordingly

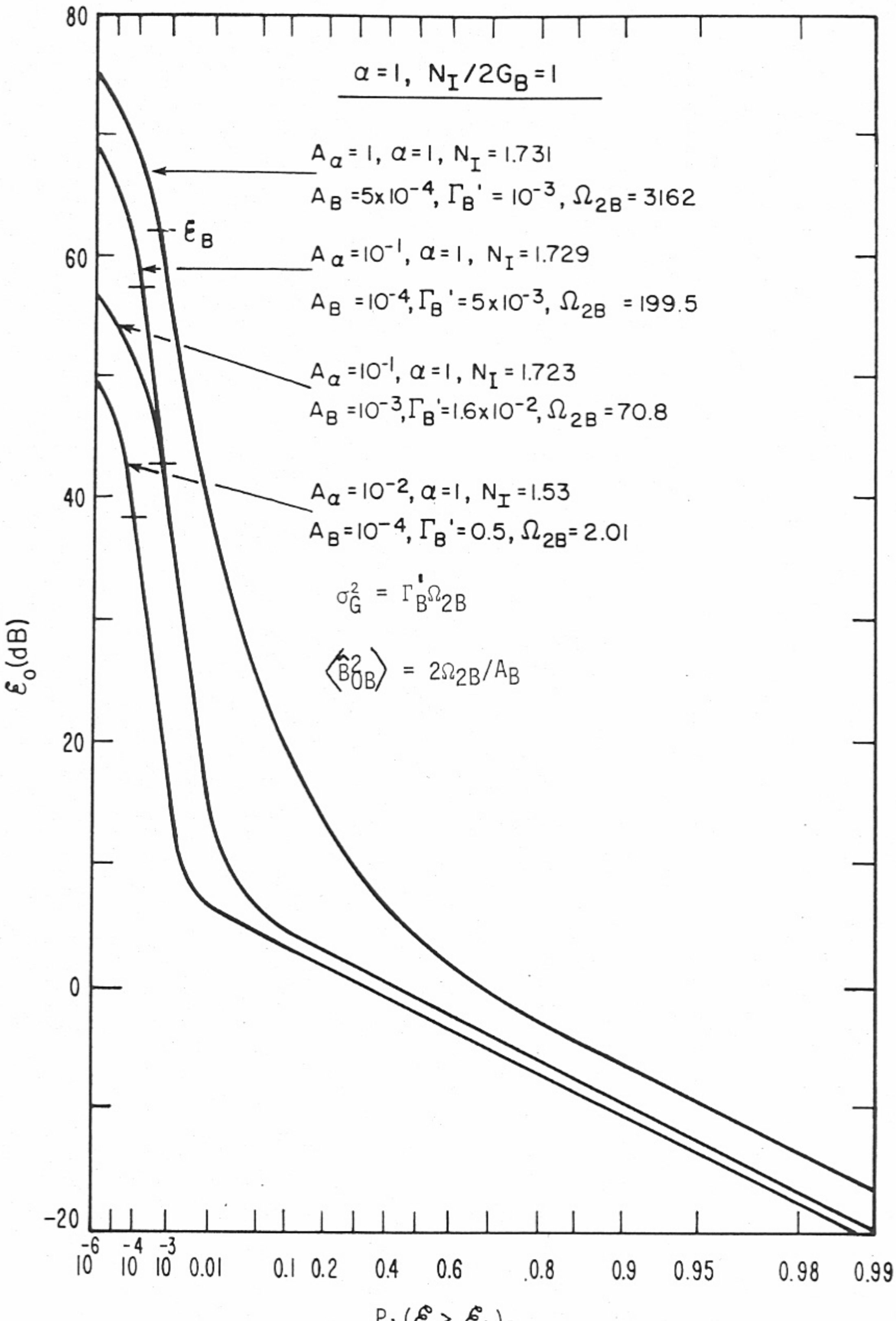

Figure 3.6 (II). The (complete) envelope distribution $P_1(\mathcal{E} > \mathcal{E}_0)_B$, eq. (3.20), calculated for Class B interference for various A_α, given α eqs. (3.11, 3.17, 3.19, 6.9, 6.10) .

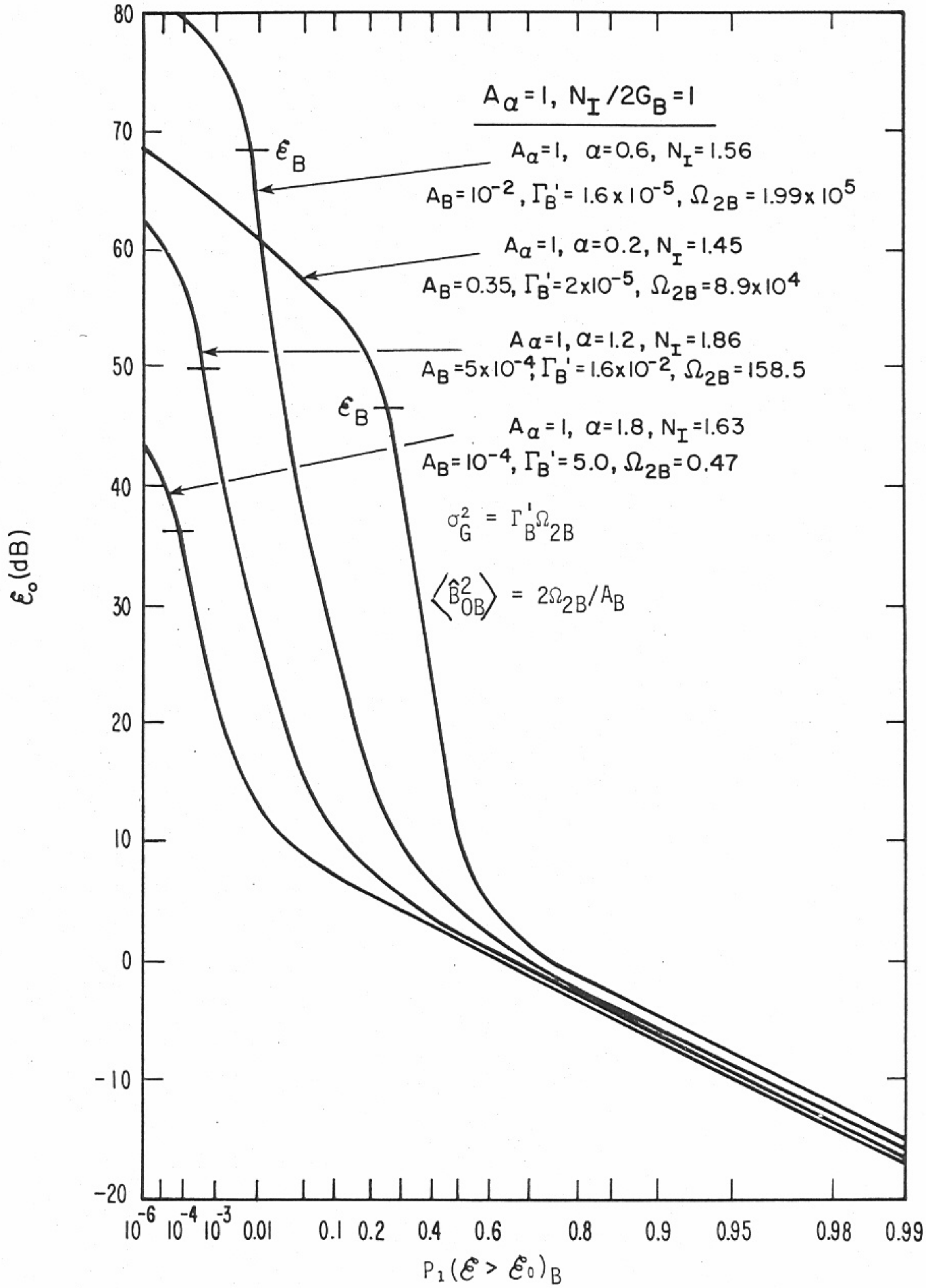

Figure 3.7 (II). The (complete) envelope distribution $P_1(\mathcal{E} > \mathcal{E}_0)_B$, eq. (3.20), calculated for Class B interference for various α, given A_α (eqs. 3.11, 3.17, 3.19, 6.9, 6.10).

obtain*

$$P_1(\mathcal{E} > \mathcal{E}_0)'_{B-I} \simeq 1-\mathcal{E}_0\int_0^\infty J_1(\mathcal{E}_0\lambda)e^{-b_{1\alpha}A_Ba^\alpha\lambda^\alpha}d\lambda. \tag{3.21}$$

This integral may be evaluated in several ways, convenient for small or large $\mathcal{E}_0$. We start with the case convenient for the small values, and employ the following transformations:

$$\left.\begin{array}{l} B_\alpha \equiv b_{1\alpha}A_Ba^\alpha(=A_\alpha\mathcal{E}^{-\alpha}); \quad B_\alpha\lambda^\alpha = z \ ; \ \therefore \lambda = (z/B_\alpha)^{1/\alpha} \\ \qquad\qquad\qquad\qquad\qquad\qquad d\lambda = dz\ z^{(1-\alpha)/\alpha}/\alpha B_\alpha^{1/\alpha} \end{array}\right\}, \tag{3.22}$$

so that (3.21) becomes now

$$P_1(\mathcal{E}>\mathcal{E}_0)'_{B-I} \simeq 1-\frac{\mathcal{E}_0^*}{\alpha}\int_0^\infty J_1(\mathcal{E}_0^*z^{1/\alpha})z^{\frac{1-\alpha}{\alpha}}e^{-z}dz \ , \ \mathcal{E}_0^* \equiv \mathcal{E}_0/B_\alpha^{1/\alpha} \ . \tag{3.23}$$

Next, let us use the Barnes integral representation of J_1:

$$J_1(\mathcal{E}_0^*z^{1/\alpha}) = \int_\Gamma \frac{\Gamma(-s)\mathcal{E}_0^{*2s+1}}{\Gamma(s+2)2^{2s+1}}z^{(2s+1)/\alpha}\frac{ds}{2\pi i} \ , \tag{3.24}$$

cf. Eq. (13.106) and Fig. 13.22 of [Middleton, 1960], where Γ is the contour $(-\infty i+c,\ i\infty+c)$, with $c(<0)$ chosen so that the integral over z in (3.23) is convergent at $z = 0$, e.g.

$$\int_0^\infty z^{(2+2s-\alpha)/\alpha}e^{-z}dz = \Gamma\left(\frac{2+2s}{\alpha}\right) \ , \ \mathrm{Re}(s) > -1, \ \therefore -1 < c < 0. \tag{3.25}$$

---

* Equation (3.21) was obtained earlier by Giordano [1970, Eq. 3.66 therein; Giordano and Haber, 1972, Eq. 24] but was not analytically evaluated. Moreover, $P_{1_{B-I}}$, here (as well as the earlier forms [Giordano, 1970], etc.) is not scaled, e.g., $\langle\mathcal{E}^2\rangle_{B-I} \neq 1$, as discussed at the beginning of A. above. See comments at end of B here.

Thus we find that (3.23) becomes

$$P_1(\mathcal{E}>\mathcal{E}_0)'_{B-I} \simeq 1-\frac{\mathcal{E}_0^*}{\alpha}\int_\Gamma \frac{\Gamma(-s)}{\Gamma(s+2)}\,\Gamma\left(\frac{2+2s}{\alpha}\right)\left(\frac{\mathcal{E}_0^*}{2}\right)^{2s+1}\frac{ds}{2\pi i} \tag{3.26a}$$

$$\simeq 1-\frac{2\mathcal{E}_0^2}{\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma\left(\frac{2+2n}{\alpha}\right)\mathcal{E}_0^{2n}}{n!(n+1)!A_\alpha^{(2+2n)/\alpha}}\quad , \tag{3.26b}$$

this last on evaluating (3.26a) at the simple poles $s = n = 0,1,2,\ldots$ and using $B_\alpha = A_\alpha 2^{-\alpha}$, cf. (3.22). Equation (3.26b) exhibits the characteristic rayleigh form ($\sim\mathcal{E}_0^2$), when $\mathcal{E}_0^2 \ll 1$, as expected.

Next, for large values of $\mathcal{E}_0$ (or small values of A_α), we return to (3.21) and use the Barnes integral representation for $\exp(-B_\alpha\lambda^\alpha)$, viz:

$$e^{-B_\alpha\lambda^\alpha} = \int_\Gamma \Gamma(-s)B_\alpha^s\lambda^{\alpha s}\frac{ds}{2\pi i}\quad , \tag{3.27}$$

to reëxpress (3.21) as

$$\left.\begin{aligned} P_1(\mathcal{E}>\mathcal{E}_0)'_{B-I} \simeq 1-\int_\Gamma \mathcal{E}_0^{*-\alpha s}\Gamma(-s)\frac{ds}{2\pi i}\int_0^\infty z^{\alpha s}J_1(z)dz, \quad -2 < \mathrm{Re}(\alpha s) < 0, \\ \mathrm{Re}(s) > -1. \end{aligned}\right\} \tag{3.28}$$

To evaluate the z-integral we use [Watson, 1944,p. 391]:

$$\int_0^\infty J_\nu(t)t^{\mu-\nu-1}dt = \Gamma(\mu/2)/\Gamma(\nu-\mu/2+1)2^{\nu-\mu+1},\ \mathrm{Re}(\mu)<\mathrm{Re}(\nu+3/2) \tag{3.29}$$

with $\nu=1, \mu = s+2$ here, and $\therefore 0<\mathrm{Re}(\alpha s+2)<2<\mathrm{Re}\ 5/2$, as required. Equation (3.28) becomes

$$P_1(\mathcal{E}>\mathcal{E}_0)'_{B-I} \simeq 1-\int_\Gamma \frac{\Gamma(-s)\Gamma(1+\frac{\alpha s}{2})}{\Gamma(1-\frac{\alpha s}{2})}2^{\alpha s}\mathcal{E}_0^{*-\alpha s}\frac{ds}{2\pi i} \tag{3.30a}$$

$$\simeq \sum_{n=1}\frac{\Gamma(1+\frac{\alpha n}{2})\ (-1)^{n+1}A_\alpha^n}{\Gamma(1-\frac{\alpha n}{2})n!\,\mathcal{E}_0^{\alpha n}}\quad , \tag{3.30b}$$

which shows how this approximation behaves as $\mathcal{E}_0 \to \infty$, viz. $0(\mathcal{E}_0^{-\alpha})$, e.g.:

$$P_1(\mathcal{E}>\mathcal{E}_0)'_{B-I} \simeq \frac{\Gamma(1+\alpha/2)A_\alpha}{\Gamma(1-\alpha/2)\mathcal{E}_0^\alpha} - \frac{\Gamma(1+\alpha)}{2\Gamma(1-\alpha)} \frac{A_\alpha^2}{\mathcal{E}_0^{2\alpha}} + 0(\mathcal{E}_0^{-3\alpha}) \ , \ 0 < \alpha < 2. \tag{3.31}$$

Now in the special case $\alpha = 1$, we may sum the series (3.26b) or (3.30b). Choosing (3.26b) we get

$$P_1(\mathcal{E}>\mathcal{E}_0)'_{B-I} \simeq 1-2\mathcal{E}_0^2 \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(2+2n)\mathcal{E}_0^{2n}}{n!(n+1)!A_1^{2+2n}} \ , \tag{3.32a}$$

and since $(n+1)! = (2)_n$, $\Gamma(2+2n) = 2^{2n}(1)_n(3/2)_n$ [Middleton, 1960, (A.1-46b), p. 1078], we find that

$$P_1(\mathcal{E}>\mathcal{E}_0)'_{B-I} \simeq 1-2\frac{\mathcal{E}_0^2}{A_1^2}\ {}_2F_1(1,3/2;2;-2\mathcal{E}_0^2/A_1^2). \tag{3.32b}$$

From [Middleton, 1960, Eq. (A.1-40c)], the (gaussian) hypergeometric function in (3.32b) is explicitly

$${}_2F_1(1,3/2;2;-x^2) = (2/-x^2)(1+x^2)^{-1/2}\left\{1-\sqrt{1+x^2}\right\},$$

so that (3.32b) reduces explicitly to

$$\boxed{P_1(\mathcal{E}>\mathcal{E}_0)'_{B-I} \simeq \frac{1}{\sqrt{1+(2\mathcal{E}_0/A_1)^2}}, \quad (\alpha = 1)} \ , \tag{3.33}$$

which is a special case of the Hall model ($\theta_{Hall} = 2$), for envelopes [Spaulding and Middleton, 1975, Eq. (2.33)]. [Note that $P_1' \simeq \mathcal{E}_0^{-1}$, $\mathcal{E}_0 \to \infty$, which checks with (3.31): in fact, both (3.31), (3.33) give $P_1' \simeq A_1/2\mathcal{E}_0$, as expected. Observe also that $P_1'(\mathcal{E}\geq\mathcal{E}_0=0)'_{B-I} = 1$, as required: P_1' is a proper P.D., although it is an inappropriate approximate form when $\Delta\sigma_G$ is at all comparable to $(b_{1\alpha}A_{1\alpha})^{1/\alpha}$, cf. (3.10a); it is also not applicable for very large $\mathcal{E}_0$, as explained earlier in A of Sec. 2.7 above. In any case, Hall-type pdf's and P.D.'s are not possible for Class A interference.]

Finally, we note that although the above PD's and pdf's, Eq. (3.26b), (3.31), exhibit the correct behaviour as $\mathcal{E}_0 \to 0$, , they are not scaled (in $\mathcal{E}_0$) properly, to provide the finite mean square needed, e.g. $\langle \mathcal{E}^2 \rangle = 1$, (cf. comments at the beginning of A above). Accordingly, as for P_{1-I} above generally, cf. (3.11b) et seq., we must replace $\mathcal{E}_0$ by $\mathcal{E}_0 N_I'$, where the scaling factor N_I' is determined, along with the four other parameters $(A_\alpha, \alpha, \Gamma_B', \Omega_{2B})$ of the distribution, by the procedure outlined in Section 6C following. [For the Hall model, $\alpha = 1$ here, and there are then only four parameter values $(N_I', A_\alpha, \Gamma_B', \Omega_{2B})$ to be established.]